Math 565: Functional Analysis
Lecture 18

Weak topology on a normed vector space.

Let X be a normed vector space, so X^* is also a normed vector space. The weak topology on X is the one generated by the functions $x \mapsto f(x) : X \to \mathbb{C}$, for $f \in X^*$. In other words, the weak top on X is generated by the sets $f^{-1}(V)$ where $f \in X^*$ and $V \subseteq \mathbb{C}$ open, so the sets $[f_1 \mapsto V_1, f_2 \mapsto V_2, \dots, f_n \mapsto V_n] := \bigcap_{i=1}^{n} f_i^{-1}(V_i)$ form a basis for the weak top. on X. Furthermore, a net $(x_i)_{i \in I}$ converges weakly, denoted $x_i \to_w x$, if $f(x_i) \to f(x)$ for all $f \in X^*$.

Recall that by Hahn-Banach, $X \cong \hat{X} \subseteq X^{**}$ via $x \mapsto \hat{x}$ is an isometry.

Obs. (a) The weak top on X is the same (homeomorphic) as the weak* top on $\hat{X} \subseteq X^{**}$.
(b) Weak top $\subseteq$ norm top on X because the functionals $f \in X^*$ are by def cont. wrt norm.
(c) Weak top is Hausdorff because weak* on $\hat{X}$ is (or more directly, we showed via Hahn-Banach that functionals in X^* separate points).
(d) On X^*, weak* $\subseteq$ weak because $\hat{X} \subseteq X^{**}$.

Prop. For a finite-dimensional normed vector space X, weak top = norm top on X, and weak* = weak = norm on X^*.

Proof. HW

Prop. Every nonempty weakly-open set contains a coset of a subspace of finite codimension.

Proof. Suffices to prove this for a basic open set $[f_1 \mapsto V_1, f_2 \mapsto V_2, \dots, f_n \mapsto V_n]$. Note that $[f_1 \mapsto V_1, f_2 \mapsto V_2, \dots, f_n \mapsto V_n] = T^{-1}(V_1 \times V_2 \times \dots \times V_n)$, where

$$T : X \to \mathbb{C}^n \text{ by } x \mapsto (f_1(x), f_2(x), \dots, f_n(x)).$$

But $\ker T$ has codim $\leq n$ since $T(X) \subseteq \mathbb{C}^n$, and $T^{-1}(V_1 \times V_2 \times \ldots \times V_n)$ is a (typically unctbl) union of cosets of $\ker T$. $\square$

Cor. For an infinite-dim normed v.s. X, every weakly open set is norm-unbdd. In particular, weak $\subsetneq$ norm top.

Proof. By the previous proposition, every weakly open set contains a coset of a finite codim subspace of X, which is hence ∞-dim (in particular, nontrivial), thus unbdd. While norm top has bdd open sets, e.g. open norm-balls. $\square$

Prop. Let X be a normed v.s.

(a) Every norm-closed subspace $Y \subseteq X$ is also weakly closed.

(b) Every norm-closed ball $B \subseteq X$ is also weakly closed.

Proof. (a) We show that $X \setminus Y$ is weakly open. Fix $x \in X \setminus Y$ and by Hahn-Banach get $f \in X^*$ s.t. $f|_Y \equiv 0$, $\|f\| = 1$, and $f(x) = \|x + Y\| =: r$. Then the weakly basic open set $[f \mapsto B_r^{\mathbb{C}}(f(x))]$ is disjoint from Y and contains x.

(b) Similar, left as HW. $\square$

Weak* vs weak top on X^*.

When a normed v.s. X is ∞-dim, so is X^* by Hahn-Banach (since otherwise, X^{**} would be fin. dim. and $X \hookrightarrow X^{**}$), hence weak* $\subseteq$ weak $\subsetneq$ norm on X^*. Also note that when X is reflexive, then weak* = weak top. on X^* because $\hat{X} = X^{**}$. We will see that the converse is true as well for all Banach spaces. For this we need the following theorem.

Theorem For any normed v.s. X, the subspace $\hat{X} \subseteq X^{**}$ is weak* dense in X^{**}.

In fact, the norm-closed unit ball of $\hat{X}$ is weak* dense in the norm-closed unit ball of X^{**}.

Proof. HW.

Cor. A normed v.s. X is reflexive $\iff$ the closed unit ball in X is weakly compact.

Proof. $\Rightarrow$. If $\hat{X} = X^{**}$, then the closed unit ball in X equipped with the weak top is homeomorphic to the closed unit ball in X^{**} equipped with the weak* top, and the latter is compact by Banach-Alaoglu.

$\Leftarrow$. If $\bar{B}_1^X$ is weakly compact, then $\bar{B}_1^{\hat{X}}$ is weak* compact, hence weak* closed since weak* top on X^{**} is Hausdorff. But by the previous thm, the weak* closure of $\bar{B}_1^{\hat{X}}$ contains $\bar{B}_1^{X^{**}}$, so $\bar{B}_1^{X^{**}} = \bar{B}_1^{\hat{X}}$, and $X^{**} = \hat{X}$ by scaling, so X is reflexive. $\square$

Cor. (a) Closed subspaces of reflexive normed vector spaces are reflexive.

(b) A Banach space X is reflexive $\iff$ X^* is reflexive.

Proof. (a) Let $Y \leq X$ be a closed subspace and X reflexive. Then Y is weakly closed by above and $\bar{B}_1^X$ is weakly compact. Thus, $\bar{B}_1^Y = \bar{B}_1^X \cap Y$ is still compact, hence Y is reflexive by the above characterization.

(b) $\Rightarrow$. If X is reflexive so $\hat{X} = X^{**}$, then $X^{***} = (\hat{X})^* = \widehat{(X^*)}$.

$\Leftarrow$. Suppose that X^* is reflexive. By (b)$\Rightarrow$, X^{**} is reflexive and $X \cong \hat{X} \leq X^{**}$ is a norm-closed subspace of X^{**} by completeness, so by (a), $\hat{X} \cong X$ is also reflexive. $\square$

Theorem. For a Banach space X, TFAE:

(1) X is reflexive.

(2) The closed unit ball in X is weakly compact.

(3) Weak* = weak top on X^*.

(4) X^* is reflexive.

Proof. We only need to prove:

(1) $\Rightarrow$ (3). Mentioned above, follows from $\hat{X} = X^{**}$.

(3) $\Rightarrow$ (4). By Banach–Alaoglu, $\overline{B}_1^{X^*}$ is weak* compact, hence weakly compact by (3), so by the characterization of reflexivity above applied to X^*, X^* is reflexive. $\square$

Strong and weak topologies on $B(X,Y)$.

For normed vector spaces X, Y, recall that $B(X,Y)$ is a normed vector space.

We consider weaker top. on $B(X,Y)$:

- the **strong top.** on $B(X,Y)$ is the one generated by the functions $T \mapsto Tx : B(X,Y) \to Y$, where $x \in X$ and Y is equipped with norm top, i.e. a net $(T_i)_{i \in I}$ converges strongly to T, written $T_i \to_s T$ $\Leftrightarrow$ $T_i x \to Tx$ in norm for all $x \in X$.
- the **weak top** on $B(X,Y)$ is the one generated by the functions $T \mapsto Tx : B(X,Y) \to Y$, where $x \in X$ and Y is equipped with weak topology, i.e. a net $(T_i) \subseteq B(X,Y)$ converges weakly to T, written $T_i \to_w T$ $\Leftrightarrow$ $g(T_i x) \to g(Tx)$ for all $x \in X$ and $g \in Y^*$.

Thus both are pointwise convergence topologies on $B(X,Y)$, where in the first case Y is equipped with the norm top., while in the second case, Y is equipped with the weak top.